# A Microeconomic Approach to Distance-Deterrence Functions in Modeling Journeys to Work

PAUL GLENN[1], INGE THORSEN[1], AND JAN UBØE[2]

[1]Stord/Haugesund College, Bjørnsonsgate 45, N-5528, Haugesund, Norway

[2]Norwegian School of Economics and Business Administration
Helleveien 30, N-5045 Bergen, Norway.

**ABSTRACT.** In this paper we suggest a microeconomic model for how commuting flows relate to traveling distance. Commuting is the preferred choice of a worker whenever he can obtain an increase in wages greater than the cost of commuting. Our framework is based on an approach where workers apply for jobs according to a strategy that maximizes their expected payoffs (wages minus commuting costs). We also discuss systematic bias in comparing the distances between city centers with the actual average traveling difference between intrazonal and interzonal commuting.

*JEL classification:* R12

*Key words:* Commuting, deterrence functions, wage differences.

## 1. Introduction

According to Griffith and Jones (1980, p. 190), spatial interaction is defined as geometric linkages between areal units over which forces pulsate that bring about spatial interrelationships. The level of spatial interaction is influenced by the fact that the areal units are geographically separated. Physical distance is the most obvious measure of separation that can be expected to influence spatial interaction. Another measure that can be relevant is the number of intervening opportunities between specific areal units. In general, one can argue that interaction flows are affected by several characteristics of spatial structure. In this paper we focus on the effect of physical distance between the points of origin and destination in space. To keep the analysis free from disturbing elements we only consider geographies with two areal units or central places. The basic mechanisms underlying the analysis are of course also relevant in systems with more complex configurations of central places.

Studies of spatial interaction and travel choices have attracted a lot of attention in transportation economics as well as in urban and regional economics and planning. Such studies constitute the basic ingredients in economic assessments of investments in transportation infrastructure. For example, models of travel choices contribute with estimates of the willingness to pay for changes in the road network, and by predicting impacts on traffic flows from the introduction of road pricing.

Models of travel choices can be distinguished according to the purpose of the study and to the level of the decision process. In general, four main categories of studies and models appear in the literature. The specific categories are trip distribution, trip generation, mode choice and trip assignment. For a standard textbook presentation of travel demand models, see Ortuzar and Willumsen (1994). For each category of models one can distinguish between several trip purposes, such as trips for shopping, recreation, or commuting to work. In this paper we consider the trip generation and the trip distribution aspects of commuting to work.

Consider the following expression of a spatial interaction model:

$$T_{ij} = f_i(A_i) f_j(B_j) C_{ij} \tag{1.1}$$

Here, $T_{ij}$ represents traffic flows from origin $i$ to destination $j$, $f_i(A_i)$ represents compound origin generativity factors and $f_j(B_j)$ represents compound destination attraction factors. When commuting flows are considered the generativity factors are related to the concentration of workers residing in zone $i$, while the attraction factors are related to the number of jobs in zone $j$. $C_{ij}$ is introduced as a generalized friction effect, representing the disutility of travel. This formulation subsumes a set of spatial interaction models. Models belonging to the gravity modeling tradition are most commonly applied to the problems relevant in this paper. For a discussion of the theoretical foundation and practical applications of gravity models, see Erlander and Stewart (1990) or Sen and Smith (1995).

In the literature on spatial interaction problems a lot of interest has been focused on the specification of the friction effect, i.e., the distance-deterrence function. For many studies this is a matter of choosing a functional specification for how interaction is deterred by physical distance. Two approaches dominate in the literature. One is the power deterrence function, defined by $C(d_{ij}) = d_{ij}^{\beta}$, the other is the exponential deterrence function, $C(d_{ij}) = \exp(-\beta d_{ij})$. For both specifications the parameter $\beta$ represents the sensitivity of the interaction volume with respect to $d_{ij}$, the physical distance between points $i$ and $j$.

One reason why the exponential distance-deterrence parameter has been widely used in empirical spatial interaction analysis is that it follows as a result of a standard derivation of the gravity model from an optimizing framework. The so-called entropy-maximizing procedure was introduced by Wilson (1967) as a theoretical basis for gravity models. This procedure results in a negative exponential impedance function. As pointed out by Ortuzar and Willumsen (1994), among others, however, reasonable reformulations of the optimizing problem result in a power function specification. It is also well known in the literature on spatial choice behavior that that traditional gravity models can be derived from stochastic utility maximization, (see for example Anas, 1983). This derivation results in an exponential distance-deterrence function if (indirect) utility is related to distance, while a power function results if it is the natural logarithm of distance that appears in the (indirect) utility function. In general, the literature offers no strong theoretical arguments in favor of one particular specification. For example, Nijkamp and Reggiani (1992) claim that the choice of a deterrence function is essentially a pragmatic one. They do, however, underline that the form of the function is strongly influenced by the spatial configuration leading to the interaction. According to Nijkamp and Reggiani (1992) the exponential structure is suitable for systems without strong barriers or metropolitan areas. Fotheringham (1983) refers to empirical support to show that a power function represents a more accurate description of the perception of distance for data on an interurban scale.

An alternative approach that has been chosen in some recent studies, is to let the data decide the specification of the distance-deterrence function. This can be achieved through a Box-Cox transformation of the distance variable. Fik and Mulligan (1998) concluded that the appropriateness of the functional form should be critically examined. Based on migration data they found that Box-Cox transformations significantly improved the goodness of fit of their gravity-related model formulation. They also found that the functional form they started out with was significantly incorrect, and that parameter estimates were considerably changed when the functional misspecification was corrected. Similar results were found in Gitlesen and Thorsen (1999) in a study of commuting flows.

In Thorsen et al. (1999) the distance-deterrence function for commuting is represented by a logistic specification. In the paper the authors introduced models defined in terms of the extreme states of the system. The focus in our paper is not particularly on these models as such. Our point of view, however, is that Thorsen et al. (1999) sets up a common framework for the discussion, comparison and visualization of any kind of model within the field. Ubøe (2001) uses this framework to discuss properties and relationships between several different kinds of gravity models. The same set of ideas will be used here to visualize the responsiveness to distance of micro-economic models for commuting.

Thorsen and Gitlesen (1998) considered the performance of gravity models to explain commuting flows. One result is that model performance improves significantly when a parameter is introduced in the distance-deterrence function that represents an additive constant term attached to the diagonal elements of the trip-distribution matrix. One possible interpretation for this is that measurement errors are introduced through the specification of distances in a discrete geography. Distances between zones are measured relative to the zonal centers, while the jobs and the residents are in general more evenly scattered over the region. Consequently the measured distance might diverge from the real distance of a journey to work.

A short survey of the extreme state approach is given in Section 2 of this paper. In Section 3 we study the determination of commuting flows through a game where wage differences are evaluated against commuting costs. In Section 4 we restrict the discussion to the cases where each individual worker is left with two or three job alternatives. In Section 5 we extend the discussion in Section 4 to the case where the labor force is divided into a number of different categories. Systematic bias due to measurement errors and a geometrical correction procedure for this are considered in Section 6. We consider the aggregate effect of labor market conditions and geometrical corrections in Section 7. Finally in Section 8 we offer some concluding remarks.

## 2. Extreme states and distance deterrence

We now give a short survey of the discussion in Thorsen et al. (1999): Consider a framework with two towns. Let $L_1, L_2$ denote the number of workers residing in towns 1 and 2, respectively, and let $E_1, E_2$ denote the corresponding number of jobs in each town. All workers are assumed to have a job, so $L_1 + L_2 = E_1 + E_2$. The trip distribution matrix $\mathbf{T}$ is then defined as $\mathbf{T} = \{T_{ij}\}_{i,j=1}^{2}$ where

(2.1)

$$T_{ij} = \text{Traffic flow, i.e., the number of people commuting from town } i \text{ to town } j$$

Thorsen et al. (1999) focus on two extreme situations:

- When commuting in the system is determined by random choice only, the expected trip distribution matrix can be expressed as follows:

$$\mathbf{T}^{\text{random}} = \begin{bmatrix} \frac{L_1 E_1}{E_1+E_2} & \frac{L_1 E_2}{E_1+E_2} \\ \frac{L_2 E_1}{E_1+E_2} & \frac{L_2 E_2}{E_1+E_2} \end{bmatrix} \tag{2.2}$$

- If, on the other hand, we consider a situation where the total traveling cost is as low as possible, we get

$$\mathbf{T}^{\text{minimal cost}} = \begin{bmatrix} \min[L_1, E_1] & L_1 - \min[L_1, E_1] \\ L_2 - \min[L_2, E_2] & \min[L_2, E_2] \end{bmatrix} \tag{2.3}$$

The basic idea in Thorsen et al. (1999) is then to write any trip-distribution matrix as a convex combination of the two extremes, i.e.,

$$\mathbf{T} = \mathbf{T}^{\text{random}}(1 - D) + \mathbf{T}^{\text{minimal cost}} D \tag{2.4}$$

$D$ measures the level of deterrence from the random cost case, and the basic hypothesis is that $D = D(d)$ where $d$ is the traveling distance. As argued in Thorsen et al. (1999), we expect that commuting will be random when the distance is very short, i.e., there is no deterrence and $D(d) \approx 0$. When the traveling distance is very long, however, we expect that action is taken so as to minimize the traveling cost in the system, i.e., there is full deterrence and $D(d) \approx 1$. To allow for some friction in the system, we consider a marginal level of interaction, $\alpha$. The distance $d_0$ signifies the distance at which $D$ is marginally close to no deterrence, and $d_\infty$ signifies the distance at which $D$ is marginally close to full deterrence. The conditions above can then be expressed as follows:

- $d \geq d_\infty \Rightarrow D(d) \geq 1 - \alpha$
- $d \leq d_0 \Rightarrow D(d) \leq \alpha$

In Figure 1, we show a distance-deterrence function corresponding to the discussion above. In Figure 1, $d_0 = 10$ (km) and $d_\infty = 60$ (km). In the regions where $d_0 \leq d$ and $d \leq d_\infty$, the function falls within $\alpha = 5\%$ of its extreme values.

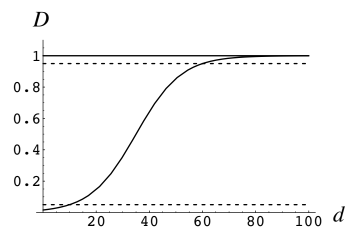


FIGURE 1: A distance-deterrence function

The main purpose of this paper is to provide a theoretical discussion of the functional form of such distance-deterrence functions. In particular, we will demonstrate that the basic properties mentioned above can be argued to be the result of a framework where the level of commuting is determined from game theory together with a geometric correction procedure.

## 3. A game theoretical approach to commuting

We now consider a situation where there are $N_1$ different types of jobs in town 1, each of which have $E_{1j}, j = 1, \ldots, N_1$ employment positions, with wages $W_{1j}, j = 1, \ldots, N_1$. Correspondingly there are $N_2$ different types of jobs in town 2, with $E_{2j}, j = 1, \ldots, N_2$ employment positions, and wages $W_{2j}, j = 1, \ldots, N_2$. The workers in each town apply for the positions with the purpose of maximizing their individual wages net of commuting costs. We assume that the jobs are sorted with the $N_1$ jobs in town 1 first, then followed by the $N_2$ jobs in town 2. The game is played as follows:

- The workers in each town hand in a set of applications for the jobs in the system. Each worker applies for one and only one job.
- If there are more applicants than jobs, the individuals getting the jobs are determined by random choice. We hence assume that all workers are equally qualified for the jobs, and invoke the law of large number to compute the distribution between towns. These jobs and the corresponding numbers of workers are then removed from the game.
- When the number of applications is less than the number of employment positions, all the applicants get hired, and the corresponding number of employment opportunities is removed from the game.

- The game is then repeated until all positions are resolved.

Since $E_1 + E_2 = L_1 + L_2$, at least one type of job will be resolved in each round, so the maximum duration of the game will be $N_1 + N_2$ rounds. The point of importance here is that workers appointed to jobs in the opposite town must adjust their wages with respect to the traveling cost, TC. If $L_j^{(i)}$ denotes the number of workers residing in town $i$ appointed to job $j$, then the average outcome of the game can be computed as follows:

$$
\begin{aligned}
&\text{Average outcome for town 1} := O_1 = \sum_{j=1}^{N_1} L_j^{(1)} W_{1j} + \sum_{j=1}^{N_2} L_{j+N_1}^{(1)} (W_{2j} - \mathrm{TC}) \\
&\text{Average outcome for town 2} := O_2 = \sum_{j=1}^{N_1} L_j^{(2)} (W_{1j} - \mathrm{TC}) + \sum_{j=1}^{N_2} L_{j+N_1}^{(2)} W_{2j}
\end{aligned}
\tag{3.1}
$$

To simplify the notation, we define $N = N_1 + N_2$, and let $W_j^{(i)}$ represent net wages for workers in town $i$ and jobtype $j$:

$$
\begin{aligned}
W_j^{(1)} &= \begin{cases} W_{1j} & \text{if } 1 \le j \le N_1 \\ (W_{2j} - \mathrm{TC}) & \text{if } N_1 < j \le N \end{cases} \\
W_j^{(2)} &= \begin{cases} (W_{1j} - \mathrm{TC}) & \text{if } 1 \le j \le N_1 \\ W_{2j} & \text{if } N_1 < j \le N \end{cases}
\end{aligned}
\tag{3.2}
$$

With this notation

$$
O_1 = \sum_{j=1}^{N} L_j^{(1)} W_j^{(1)} \qquad O_2 = \sum_{j=1}^{N} L_j^{(2)} W_j^{(2)}
\tag{3.3}
$$

Note that all mixed strategies for this game can be defined from a choice probability $P$ defined on the set of all types of jobs, i.e., a function defining the probability that a worker will apply for each type of job given that his choice is restricted to a particular subset of the jobs.

The game defined above has $L_1 + L_2$ individual players. Each player in town 1 gets an average outcome equal to the average outcome for town 1 divided by $L_1$, with a corresponding result for town 2. Since $L_1$ and $L_2$ are fixed, we can assume without loss of generality that the players try to maximize the outcomes in (3.2). All the players within the same town are by assumption identical. If we make the additional assumption that each such player has a unique best strategy, and that all the players use this strategy, we may essentially consider this as a two player game; all the workers in town 1 apply the same choice probability, $P$, and all the workers in town 2 apply the same choice probability $Q$.

## 4. 2-job and 3-job games

To simplify the discussion further, we will restrict ourselves to some special cases. The simplest such case occurs when there is only one job in each town, i.e., $N_1 = N_2 = 1$. We refer to this situation as a 2-job game. In this case we let $p$ denote the fraction of workers in town 1 applying for the job in town 1, and $q$ denotes the fraction of workers in town 2 applying for the job in town 1. The outcome of the game in this case is fairly obvious, but we include some details merely as an illustration of the technical complexity of the game.

(4.1)
$$O_1[p,q] = \begin{cases} \frac{pL_1E_{11}}{pL_1+qL_2}W_1^{(1)} + (L_1 - \frac{pL_1E_{11}}{pL_1+qL_2})W_2^{(1)} & \text{if } pL_1 + qL_2 \geq E_{11} \\ (L_1 - \frac{(1-p)L_1E_{21}}{(1-p)L_1+(1-q)L_2})W_1^{(1)} + \frac{(1-p)L_1E_{21}}{(1-p)L_1+(1-q)L_2}W_2^{(1)} & \text{if } pL_1 + qL_2 < E_{11} \end{cases}$$

(4.2)
$$O_2[p,q] = \begin{cases} \frac{qL_2E_{11}}{pL_1+qL_2}W_1^{(2)} + (L_2 - \frac{qL_2E_{11}}{pL_1+qL_2})W_2^{(2)} & \text{if } pL_1 + qL_2 \geq E_{11} \\ (L_2 - \frac{(1-q)L_2E_{21}}{(1-p)L_1+(1-q)L_2})W_1^{(2)} + \frac{(1-q)L_2E_{21}}{(1-p)L_1+(1-q)L_2}W_2^{(2)} & \text{if } pL_1 + qL_2 < E_{11} \end{cases}$$

From (4.1) and (4.2), we get

$$\text{(4.3)} \qquad \frac{\partial O_1}{\partial p} = \begin{cases} \frac{qL_1L_2E_{11}}{(pL_1+qL_2)2}(W_1^{(1)} - W_2^{(1)}) & \text{if } pL_1 + qL_2 \geq E_{11} \\ \frac{(1-q)L_1L_2E_{11}}{((1-p)L_1+(1-q)L_2)2}(W_1^{(1)} - W_2^{(1)}) & \text{if } pL_1 + qL_2 < E_{11} \end{cases}$$

$$\text{(4.4)} \qquad \frac{\partial O_2}{\partial q} = \begin{cases} \frac{pL_1L_2E_{11}}{(pL_1+qL_2)2}(W_1^{(2)} - W_2^{(2)}) & \text{if } pL_1 + qL_2 \geq E_{11} \\ \frac{(1-p)L_1L_2E_{11}}{((1-p)L_1+(1-q)L_2)2}(W_1^{(2)} - W_2^{(2)}) & \text{if } pL_1 + qL_2 < E_{11} \end{cases}$$

From (4.3) and (4.4), it follows that $O_1$ is monotone in $p$, and that $O_2$ is monotone in $q$. Hence if $W_1^{(1)} \neq W_2^{(1)}$ and $W_1^{(2)} \neq W_2^{(2)}$, then there is a unique Nash equilibrium characterized by a strategy where a player always applies for the job with the best wages. If $W_1^{(1)} = W_2^{(1)}$, then player 1 is indifferent between all the strategies, and may just as well only apply for job 1. If $W_1^{(2)} = W_2^{(2)}$, then player 2 is indifferent between all the strategies, and may just as well only apply for job 2. Because of the traveling costs, preferences may be subject to change when the distance $d$ between town 1 and town 2 changes. Figure 2 shows a numerical simulation of the distance-deterrence function in this case. In Figure 2 we have used the values $E_1 =$

$11\,423, E_2 = 7\,577, L_1 = 13\,276, L_2 = 5\,724, W_{11} = 30\,000, W_{21} = 32\,000$ subject to a traveling cost of \$50 per kilometer per year.

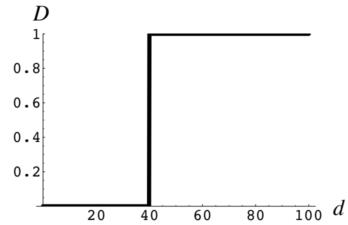


FIGURE 2: A distance-deterrence function for a 2-job game

Now consider the case where there are two different jobs in town 1 and one job in town 2, i.e., $N_1 = 2$ and $N_2 = 1$. The outcome of the game is now quite different. The players have to balance the possibility of getting the best job against the danger of falling into the worst category. In this case the game is too complex to admit a straightforward analysis, so we will resort to a numerical approach. One basic principle remains from the 2-job game, however; the players never apply for the worst category.

EXAMPLE 4.1

In this example we set $L_1 = 4000, L_2 = 6000, E_{11} = 4000, E_{12} = 2500$ and $E_{21} = 3500$. The wages are $W_{11} = 29000$ (\$/year), $W_{12} = 22000$ (\$/year), and $W_{21} = 30000$ (\$/year). The wage differences have to be balanced against a traveling cost of 80 (\$ per km per year). From the numerical simulations, we see that the players now continuously change their strategy. The resulting distance-deterrence function is shown in Figure 3.

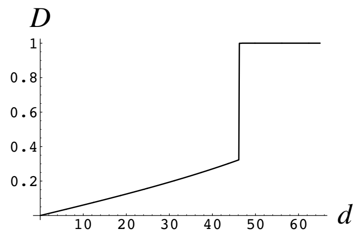


FIGURE 3: A distance-deterrence function for a 3-job game

In this case the non-trivial choice probabilities are the ones for job 1 and job 3 on the first level. The players never apply for job 2, and on the second level there are at most two jobs left, so the analysis for the 2-job game applies. We mention a few typical values: At $d = 20$ the players in town 1 apply $p_1 = 0.58$ and $p_3 = 0.42$, while the players in town 2 use $q_1 = 0.48$ and $q_3 = 0.52$. At $d = 40$ km the players in town 1 apply $p_1 = 0.67$ and $p_3 = 0.37$, while the players in town 2 use $q_1 = 0.46$ and $q_3 = 0.54$. The corresponding trip distribution matrices are as follows

$$\mathbf{T}_{20} = \begin{pmatrix} 2774 & 1226 \\ 3726 & 2274 \end{pmatrix} \qquad \mathbf{T}_{40} = \begin{pmatrix} 2978 & 1022 \\ 3522 & 2478 \end{pmatrix} \tag{4.5}$$

and, as can be expected, we observe an increase in internal commuting.

## 5. The $N$-job game with multiple categories

To proceed one step further, we consider a case where there are $M$ different job categories, and where the same job categories are found in both towns. Each worker is only qualified for work in one job category, but jobs offered for a specific category are not homogeneous. Each category (profession) consists of workers with specific qualifications, but as a group they are faced with job offers of varying tasks, responsibilities and wages.

Hence the labor force $L_1$ in town 1 can be divided into categories $L_{11}, L_{12}, \ldots, L_{1M}$, where each category is qualified for its corresponding job category. The corresponding categories in town 2 are denoted $L_{21}, L_{22}, \ldots, L_{2M}$. We will assume that the workers $L_{1k}, L_{2k}$ in each category are qualified for the jobs $E_{1jk}, k = 1, \ldots, N_{1k}$ and $E_{2jk}, k = 1, \ldots, N_{2k}$ only. All the workers within each category compete for the jobs according to the game in the previous section. By a slight abuse of notation, we define $E_{1k} = \sum_{j=1}^{N_{1j}} E_{1jk}$ and $E_{2k} = \sum_{j=1}^{N_{2j}} E_{2jk}$, i.e., the total number of employment opportunities in each town and category.

In the numerical simulations, we only consider two special cases. In the first case we assume that within each category, there is one and only one job option in each town. Hence $N_{1k} = N_{2k} = 1$ for all $k$, and $E_{1k} = E_{11k}, E_{2k} = E_{21k}$ denote the number of jobs available for category $k$. In the second case we assume $N_{1k} = 2, N_{2k} = 1$ for each $k$, in which case $E_{11k}, E_{12k}$ and $E_{21k}$ denote the number of available jobs available for category $k$. Then $E_{1k} = E_{11k} + E_{12k}$ denotes the total number of available jobs in town 1 in category $k$, and $E_{2k} = E_{21k}$ denotes the total number of available jobs in town 2 in category $k$.

When a population is divided into categories as above, the matrices $\mathbf{T}^{\text{random}}$ and $\mathbf{T}^{\text{minimal cost}}$ must be computed from the expressions

$$\mathbf{T}^{\text{random}} = \begin{bmatrix} \sum_k \frac{L_{1k}E_{1k}}{E_{1k}+E_{2k}} & \sum_k \frac{L_{1k}E_{2k}}{E_{1k}+E_{2k}} \\ \sum_k \frac{L_{2k}E_{1k}}{E_{1k}+E_{2k}} & \sum_k \frac{L_{2k}E_{2k}}{E_{1k}+E_{2k}} \end{bmatrix} \tag{5.1}$$

$$\mathbf{T}^{\text{minimal cost}} = \begin{bmatrix} \sum_k \min\{L_{1k}, E_{1k}\} & L_1 - \sum_k \min\{L_{1k}, E_{1k}\} \\ L_2 - \sum_k \min\{L_{2k}, E_{2k}\} & \sum_k \min\{L_{2k}, E_{2k}\} \end{bmatrix} \tag{5.2}$$

Note that the matrices defined by (5.1) and (5.2) will in general be different from the corresponding expressions given by (2.2) and (2.3). To illustrate this, we consider the following example:

Assume that we have 2 different categories, with values

$$E_{11} = 100, E_{12} = 0, E_{21} = 0, E_{22} = 300 \tag{5.3}$$

and

$$L_{11} = 0, L_{12} = 300, L_{21} = 100, L_{22} = 0 \tag{5.4}$$

In this case we get $E_1 = 100, E_2 = 300$ and $L_1 = 300, L_2 = 100$. Using the aggregated data in (2.2) and (2.3), we get:

$$\mathbf{T}^{\text{random}} = \begin{bmatrix} 75 & 225 \\ 25 & 75 \end{bmatrix} \qquad \mathbf{T}^{\text{minimal cost}} = \begin{bmatrix} 100 & 200 \\ 0 & 100 \end{bmatrix} \tag{5.5}$$

while the correct specification is given by (5.1) and (5.2), i.e.,

$$\mathbf{T}^{\text{random}} = \begin{bmatrix} 0 & 300 \\ 100 & 0 \end{bmatrix} \qquad \mathbf{T}^{\text{minimal cost}} = \begin{bmatrix} 0 & 300 \\ 100 & 0 \end{bmatrix} \tag{5.6}$$

In this example all workers must commute to the neighboring town. Hence the aggregated data in (5.5) will give a serious misspecification of the distance deterrence. This example is extreme, but it illustrates a point of importance: in modeling commuting to work, each category of the labor force must be treated separately, and the results from each category added together.

For a throughout discussion of aggregation and the modeling issues it poses with respect to gravity models, see Ubøe (2001). Here we will need to compute distance-deterrence functions for aggregated systems, and we refer to the following result from Ubøe (2001).

THEOREM Ubøe (2001)

*Consider an aggregated system of $M$ categories, where each of the categories $k$ has a distance-deterrence function $D_k, k = 1, \ldots, M$. If we let $E_k = E_{1k} + E_{2k}, k = 1, \ldots, M$ and $L_k = L_{1k} + L_{2k}, k = 1, \ldots, M$ denote the total number of employment opportunities/workers in each category $k$ in the whole system, then the distance deterrence $D$ for the aggregated system can be found as follows:*

$$D[d] = \frac{\sum_{k=1}^{M} L_k \min\left[\frac{E_{1k}}{E_k} \cdot \frac{L_{2k}}{L_k}, \frac{E_{2k}}{E_k} \cdot \frac{L_{1k}}{L_k}\right] \cdot D_k[d]}{\sum_{k=1}^{M} L_k \min\left[\frac{E_{1k}}{E_k} \cdot \frac{L_{2k}}{L_k}, \frac{E_{2k}}{E_k} \cdot \frac{L_{1k}}{L_k}\right]} \tag{5.7}$$

When the working population is divided into several categories, we get a more complex interaction pattern for the distance-deterrence function even for the 2-job game. An example of this with 3 different categories (professions) is shown in Figure 4.

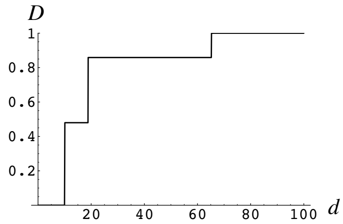


FIGURE 4:
A distance-deterrence function for a 2-job game with 3 different categories

In systems with a small number of categories, the distance-deterrence function can have almost any shape according to the characteristics of each category. As the system is refined into a large number of small categories, however, it seems reasonable to conjecture that the local anomalies will be significantly reduced. Figure 5 shows a numerical simulation of the distance-deterrence function in a system where the workers are subdivided into 500 different professions. In this framework, commuting is computed on the basis of wage differences between the two towns. In Figure 5 we have used a construction where the difference in wages decreases with the size of the categories (see the theoretical discussion below).

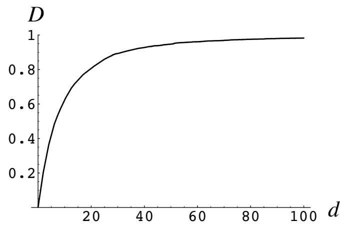


FIGURE 5:
A distance-deterrence function for a 2-job game with 500 different categories

The resulting picture in Figure 5 is a globally-concave function. To examine this further, we repeated the experiment for a 3-job game. In this case the game comes out with a variety of different curves and patterns depending on the relative size of the various groups. A typical selection is shown in Figure 6 below. When these systems are aggregated, however, all the peculiarities are wiped out, and the result is again a globally-concave function. Figure 7 below shows the result of an aggregation of 500 categories.

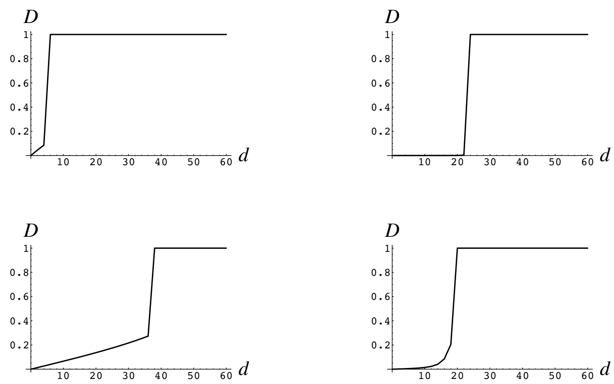


FIGURE 6: A collection of distance-deterrence functions for a 3-job game

The curves in Figure 6 and 7 have been constructed as follows: We first sampled $L_k = E_k, k = 1, \ldots, 500$ from a uniform distribution on the interval $[100, 10000]$. Then each $L_k$ was subdivided into $L_{1k}, L_{2k}$ by random choice, and correspondingly, we split $E_k$ into $E_{11k}, E_{12k}$ and $E_{21k}$. To simplify the programming, we always placed 2 jobs in town 1. Finally the wages $W_{11k}, W_{12k}$, and $W_{21k}$ were drawn from a uniform distribution on the interval $[30000 \cdot (1 - 100/L_k), 33000]$.

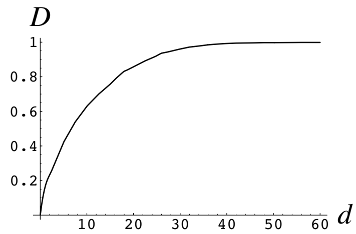


FIGURE 7:
A distance-deterrence function for a 3-job game with 500 different categories

*Theoretical discussion*

To try to explain these results from a theoretical point of view, we consider the 2-job game. We let $\Delta W_k = |W_{1k} - W_{2k}|, k = 1, \ldots, M$, i.e., the difference in wages between the two towns in category $k$. For this game the distance-deterrence functions are given by the simple expression

$$D_k[d] = \mathcal{X}_{\Delta W_k \le \mathrm{TC}[d]} = \begin{cases} 1 & \text{if } \Delta W_k \le \mathrm{TC}[d] \\ 0 & \text{otherwise} \end{cases} \tag{5.8}$$

Hence from (5.7), we get

$$D[d] = \frac{\sum_{k=1}^{M} L_k \min\left[\frac{E_{1k}}{E_k} \cdot \frac{L_{2k}}{L_k}, \frac{E_{2k}}{E_k} \cdot \frac{L_{1k}}{L_k}\right] \cdot \mathcal{X}_{\Delta W_k \le \mathrm{TC}[d]}}{\sum_{k=1}^{M} L_k \min\left[\frac{E_{1k}}{E_k} \cdot \frac{L_{2k}}{L_k}, \frac{E_{2k}}{E_k} \cdot \frac{L_{1k}}{L_k}\right]} \tag{5.9}$$

If we consider a distance where $\mathrm{TC}[d] = \Delta W_k$, the curve jumps upwards with a jump of relative size

$$L_k \min\left[\frac{E_{1k}}{E_k} \cdot \frac{L_{2k}}{L_k}, \frac{E_{2k}}{E_k} \cdot \frac{L_{1k}}{L_k}\right] \tag{5.10}$$

To explain what is happening with the curve, we fix a distance $d_0$ and consider all the job categories where $\text{TC}[d_0] \leq \Delta W_k \leq \text{TC}[d_0 + \Delta d_0]$. If $\Delta d_0$ is not too small, there will be many job categories within this range of wage differences. What controls the size of the jumps is mainly the first term in (5.10), i.e. the size of the total population, $L_k$. Consider the second term in (5.10), i.e., $\min\left[\frac{E_{1k}}{E_k} \cdot \frac{L_{2k}}{L_k}, \frac{E_{2k}}{E_k} \cdot \frac{L_{1k}}{L_k}\right]$. Note that the largest value of this term is obtained whenever

$$E_{1k} = E_{2k} = L_{1k} = L_{2k} = 0.5 \cdot E_k$$

If $E_{1k} \neq 0.5 \cdot E_k$, however, the largest value is obtained whenever $L_{1k} = E_{1k}$. Hence, the more symmetry, the more impact the term has on the final curve. This term thus adjusts for an uneven spread in the fraction of employment opportunities in a given job category between each town in relationship to the fraction of workers in the same job category. If we assume that the two stochastic terms in (5.10) are independent of each other, i.e., that the proportion of local employment in each job category is largely independent of the population size, $L_k$, then the effect of the adjustments will average out. This being so, the size of the jump in the interval $[d_0, [d_0 + \Delta d_0]]$ will be proportional to

$$\sum_{k:\text{TC}[d_0] \leq \Delta W_k \leq \text{TC}[d_0+\Delta d_0]} L_k \approx \sum_{k:\text{TC}[d_0] \leq \Delta W_k \leq \text{TC}[d_0+\Delta d_0]} \text{Expectation}[L_k] \tag{5.11}$$

invoking the law of large numbers. What happens with the average number of workers in these job categories when $d_0$ increases and $\Delta d_0$ is fixed? We think it is reasonable to assume that

- The number of job categories with $\text{TC}[d_0] \leq \Delta W_k \leq \text{TC}[d_0 + \Delta d_0]$ goes down

For example, there will be fewer job categories where the wage differences between the two towns are in the interval between \$10000 and \$10500 than in the interval between \$1000 and \$1500.

- The expected population, $\text{Expectation}[L_k]$, goes down

That is, that jobs with large wage differences are generally more specialized, consisting of a smaller number of workers.

Under these assumptions, the relative increase in $D$ will decrease with increasing $d$, giving rise to a globally concave profile. This argument applies to the 2-job game. For the $N$-job game the analysis is much more difficult. Some parts of the previous discussion applies, however. Formula (5.7) can be used in this case as well. Hence much of the behavior is controlled by the coefficient

$$L_k \min\left[\frac{E_{1k}}{E_k} \cdot \frac{L_{2k}}{L_k}, \frac{E_{2k}}{E_k} \cdot \frac{L_{1k}}{L_k}\right] \tag{5.12}$$

For the $N$-job case the particular shape of each $D_k, k = 1, \ldots, M$ is too complicated to admit a detailed analysis. From the numerical simulations shown in Figure 7, however, it seems reasonable to expect that the same principle holds for this case as well, i.e., that distance-deterrence curves for aggregated systems can be expected to be globally concave.

*Some remarks*

Ubøe (2001) considers aggregated systems of gravity models with an exponential-deterrence function, and obtains a similar result: the deterrence functions are concave at moderate and large distances and close to linear at short distances. Notice that the behavior of each category is completely different from the approach in the present paper. In Ubøe (2001) the categories behave according to a random utility maximation. In the present paper the behavior is determined from a fixed rule determined by the game. The deterrence functions for each subcategory of workers are completely different in the two cases. Nevertheless we get exactly the same kind of response in the aggregate system.

It seems quite unlikely that one could improve the specification of the curves much beyond the point of globally concave functions. Given any such properly-scaled function $f$, it is probably possible to backtrack the construction to set up an aggregated system with $D = f$. In this connection, note that an aggregated system of gravity models with an exponential-deterrence function is not, in general, of an exponential type, see (Ubøe, 2001). Hence particular functional forms such as exponential or power-function specifications generally fail in aggregate systems.

## 6. Geometric corrections

Our model takes into account the expenses for commuting between towns, and we have used the distance between the centers in each town to calculate the traveling costs. This is a reasonable approach as long as all internal traveling distances are small. In general, however, the distances between urban centers will have to be adjusted with respect to intrazonal traveling distances. Traveling distance in itself is not crucial; the important factor is the actual difference in traveling distance between the alternative job locations.

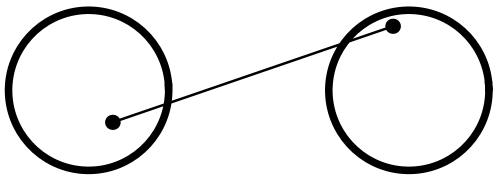

FIGURE 8: Euclidean distances

Consider the framework in Figure 8. The first town is separated from the second one by a distance $d$ between the two centers. The actual difference in traveling distance is, in general, different from $d$. To analyze this further we start out with a number of simplifying assumptions. We assume that movement is unrestricted, i.e., that any pair of positions within the two towns can be joined by a link with length equal to the euclidean distance between the two positions. We also assume that job and residential sites are uniformly distributed within two circles, each of which has radius $r$. If so, the average internal traveling distance, ITD, is given by the integral

$$\text{ITD} = \frac{1}{\pi^2 r^4}\int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\int_{-r}^{r}\int_{-\sqrt{r^2-u^2}}^{\sqrt{r^2-u^2}}\sqrt{(x-u)^2+(y-v)^2}\,dv\,du\,dy\,dx \tag{6.1}$$

The average external traveling distance, ETD = ETD($d$), is given by the corresponding expression

(6.2)

$$\text{ETD}(d) = \frac{1}{\pi^2 r^4}\int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\int_{-r}^{r}\int_{-\sqrt{r^2-u^2}}^{\sqrt{r^2-u^2}}\sqrt{(x-u+d)^2+(y-v)^2}\,dv\,du\,dy\,dx$$

If we are to compare the average difference in traveling expenses between intrazonal and interzonal commuting, we must consider the average difference in traveling distances, $G_r(d) = \text{ETD}(d) - \text{ITD}$. Figure 9 shows the graph of this function in the case where $r = 10$ (km). The straight line corresponds to the case where intrazonal distances are zero.

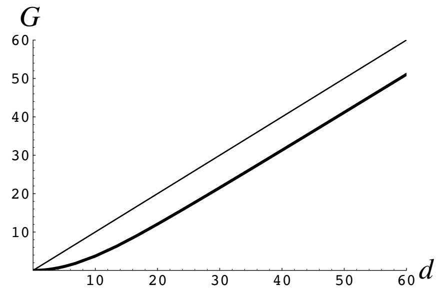


FIGURE 9: Geometric correction, euclidean case

From the graph we may conclude that the effect of intrazonal traveling is highly significant. Given that

$$\lim_{d\to\infty} d - \sqrt{(x-u+d)^2+(y-v)^2} = u - x \tag{6.3}$$

it is easy to see that $\lim_{d\to\infty} d - G_r(d) = \text{ITD}$. Hence the long range correction is equal to the average intrazonal traveling distance.

One argument against the construction above is that motion is usually restricted to movement along fixed roads, and that euclidean distance cannot be obtained. To determine the effect of this we carried out the same kind of construction for the cases shown in Figure 10.

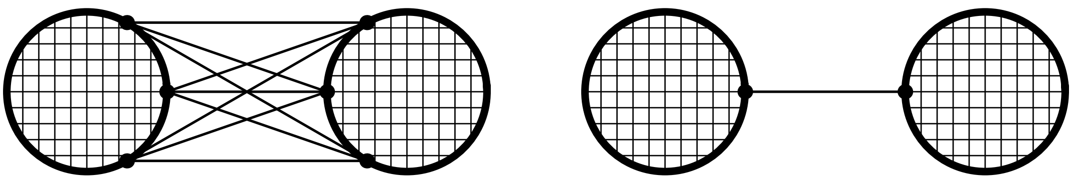

FIGURE 10: Restricted geometries

In both of these cases travel is restricted to movement along the straight lines in each figure. The resulting differences in the average traveling distance are shown in Figure 11 together with the results from the euclidean case. In Figure 11 the geometry from the left-hand side of Figure 10 is represented by the dotted line, while the geometry from the right-hand side is depicted by the dashed line.

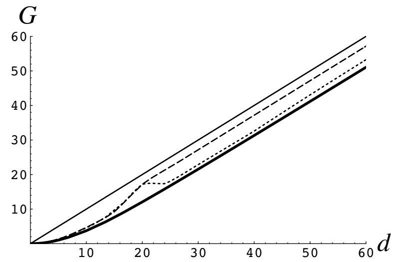


FIGURE 11: Geometric corrections

Ignoring the part of the curve near $d = 20$, it is surprising to see that the incorporation of just a few interconnecting lines pulls the solution strongly in the direction of the euclidean case. The behavior near $d = 20$ is very artificial, and is due to collapsing geometries when the two circles touch each other. This part of the curve has no practical significance. If two towns are situated in a position such as this, one would expect a number of small roads connecting the areas. This would pull the solution strongly in the direction of the euclidean case.

A geometric problem of a different kind is encountered whenever a town«s geography does not coincide with the dispersion of the actual population. Two examples of this type are shown in Figure 12.

FIGURE 12: Misspecifications of city center and extension

In the figure on the left-hand side of Figure 12, the intrazonal traveling distance will be misspecified if it is calculated with reference to the total geographical extension of the town. In the figure on the right hand side we need to adjust both the center and the extension to give an appropriate description of the population density.

## 7. Composition of distance deterrence and geometric corrections

From the arguments in Section 6, it seems reasonable to model the distance deterrence from a globally concave function depending on the actual traveling distance $x$. As an example of this , we suggest using the simple expression

$$D(x) = 1 - e^{-\gamma x} \tag{7.1}$$

where we, in the spirit of Section 2, set $\gamma = \frac{1}{d_\infty} \ln\left(\frac{1}{\alpha}\right)$. In applications of this theory, however, we usually do not refer to actual traveling distances as $d$ but rather refer to the difference between two city centers as $d$. As argued in Section 5, we quite often expect to find a systematic bias between $d$ and the average difference in traveling distance between intrazonal and interzonal traveling. From this point of view, we suggest using the euclidean correction $x = G_r(d)$. Here, $r$ is the radius of the town. In the spirit of Section 2 again, we put $r = d_0$. This gives us

$$(7.2) \qquad D(d) = D_{d_0,d_\infty}(d) = 1 - e^{-\gamma G_{d_0}(d)} \qquad \text{with } \gamma = \frac{1}{G_{d_0}(d_\infty)} \ln\left(\frac{1}{\alpha}\right)$$

A plot of this function using the parameter values $\alpha = 5\%, d_0 = 10$ and $d_\infty = 60$ is shown in Figure 13.

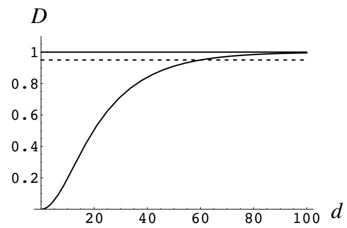


FIGURE 13: A model for distance deterrence

Thorsen et al. (1999) suggested that one could model $D(d)$ using a logistic function. This is the function shown in Figure 1. As we can see from Figure 10, the resulting graph is not very different. Nevertheless, it is our opinion that the composite structure above is more satisfying from a theoretical point of view.

## 8. Concluding remarks

In this paper we have discussed some theoretical aspects of distance-deterrence functions in modeling commuting to work. We started out by defining a game where each worker applies for a job in one of two towns in the geography. The simplest case occurs when there is only one kind of job in each town. Commuting flows are the result of a unique Nash equilibrium determined by wage differentials and the distance between the towns. A more complex situation arises in the case where there are many different job alternatives for each type of worker (profession). We base our discussion on the assumption that the individual workers are qualified for one and only one job category. We demonstrate that commuting flows in such situations result not only from wage differentials and distances, but also from a spatial mismatch between the types of jobs and the categories of workers. This represents a potential problem when distance-deterrence parameters are estimated from aggregated data on commuting flows. We formulate explicit conditions that have to be met to avoid biased estimates of parameters that are intended to measure behavioral responses to variations in distance. We then carry out simulation experiments based on the specification of the game. Finally, we discuss analytical results for the resulting distance-deterrence function in the case where the number of job categories is very large. Under quite general conditions we are able to demonstrate that the distance-deterrence function is globally concave when observed on a scale that is not too small.

The globally concave distance-deterrence function does not account for possible measurement errors resulting from a practice where distances between zones are measured relative to the zonal centers, while the jobs and the residents are in general more evenly scattered over the region. We argue that this calls for a geometric correction of distances, and demonstrate that this correction typically should correspond to the average intrazonal traveling distance. Through this geometric correction of the otherwise globally concave function we end up with a distance-deterrence function with a logistic profile.

It is of course possible to extend the game-theoretical analysis considerably in many directions from the simple framework that is offered in this paper. The main purpose of this paper is to demonstrate that it is possible to derive a profile for the relationship between distances and commuting flows from a game-theoretical specification of a labour market equilibrium. This profile can next be compared to the alternative distance-deterrence functions that appears in the literature. Considering, for example, the experiences with Box-Cox transformations that are mentioned in the introduction, we find it surprising that so little attention has been directed towards purely theoretical aspects of the distance-deterrence relationship in the literature on spatial interaction problems.

## REFERENCES


Anas, A. (1983) Discrete choice theory, information theory and the multinomial logit and gravity models. *Transportation Research B*, 17: 13–23

Erlander, S. and N.F. Stewart (1990) *The gravity model in transportation analysis - theory and extensions*, VSP, Utrecht

Fik T.J. and G.F, Mulligan (1998) Functional form and spatial interaction models. Environment and Planning A, volume 30: 1497–1507

Fotheringham, A.S. (1983) A new set of spatial-interaction models: the theory of competing destinations. *Environment and Planning A*, volume 15: 15–36

Fotheringham, A.S. (1981) Spatial structure and distance-decay parameters. *Annals of the Association of American Geographers*, vol. 71: 425–436

Fotheringham, A.S. (1984) Some theoretical aspects of destination choice and their relevance to production-constrained gravity models. *Environment and Planning A*, volume 15: 1121–1132

Gitlesen, J.P. and I. Thorsen (1999) An empirical evaluation of how commuting flows respond to new road connections and toll charges, paper presented Svinkløv, Denmark, at the 4th workshop in the Nordic Research Network on Modeling Transport, Land-Use and the Environment

Griffith, D.A. and K. Jones (1980) Explorations in the relationship between spatial structure and spatial interaction. *Environment and Planning A*, 12: 187–201

Nijkamp, P. and A. Reggiani (1992) *Interaction, evolution and chaos in space*, Springer-Verlag

Ortuzar, J. de D. and L. G. Willumsen (1994) *Modeling transport*. John Wiley & Sons Ltd

Thorsen, I. and J.P. Gitlesen (1998) Empirical evaluation of alternative model specifications to predict commuting flows. *Journal of Regional Science*, Vol. 38, 2: 273–292

Thorsen, I., J. Ubøe and G. Nævdal (1999) A Network Approach to Commuting. *Journal of Regional Science*, vol. 39, No.1: 73–101

Sen, A. and T. Smith (1995) *Gravity models of spatial interaction behaviour*, Springer-Verlag, Berlin Heidelberg

Ubøe, J. (2001) Aggregation of Gravity Models for Journeys-to-work, working paper

Wilson, A.G. (1967) A statistical theory of spatial distribution models. *Transportation Research* 1: 253–269.